Mini-Course 4: Matrix Completion has No Spurious Local Minimum

Yang Yuan

Computer Science Department
Cornell University

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  - Every local minimum is close to one global minimum.
- Therefore, all local minima are global minima

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  - $\|Z_i\|^2 = (ZZ^\top)_{i,i} = M_{i,i}$

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- **Linear properties** are preserved by concentration inequality:

$$\sum_{(i,j)\in\Omega} M_{i,j} \approx p \times \sum_{(i,j)\in d\times d} M_{i,j}$$

Concentration inequalities for linear properties

Theorem (Random sampling$\rightarrow$ good spectral approximation)

If $\|M\|_\infty \leq \frac{\|M\|_F}{d}$, *whp,*

$$\|P_\Omega(M) - pM\| \leq \epsilon p \|M\|_F$$

where $\epsilon = O(\sqrt{\log d/(pd)})$.

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Proof.

- $P_\Omega(M) = \sum_{i,j} s_{i,j} M_{i,j} \delta_{i,j}$, Where $\delta_{i,j} \in \mathbb{R}^{d\times d}$ is indicator matrix, $s_{i,j} \in \{0, 1\}$ are Bernoulli variable with probability p.

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- Apply Bernstein inequality. □

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  - $\langle M, Z\rangle \triangleq \mathrm{tr}(M^\top Z)$, entry-wise inner product.

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- $\|P_\Omega(XX^\top)X - pXX^\top X\|_F \le p\epsilon\|X\|_F^3$

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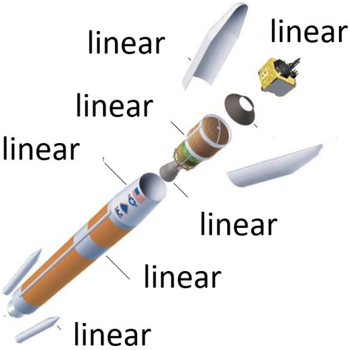

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- Assume $M = zz^\top$.

$$f(x) = \frac{1}{2}\|P_\Omega(M - xx^\top)\|_F^2 + \lambda R(x)$$

where $R(x) = \sum_{i=1}^{d}(|x_i| - \alpha)^4 \mathbb{1}_{|x_i| \geq \alpha}$. $\alpha = \frac{10\mu}{\sqrt{d}}$.

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  - Add $R(x)$ to make sure x couldn't have large coordinates.

Proof sketch

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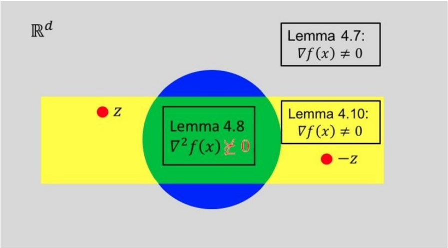
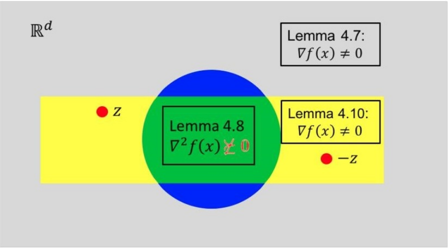


Proof sketch

$\nabla f(x) = 0$ implies

- $\|x\|_\infty \leq 4\alpha$. i.e., inside yellow.
- $\|zz^\top x - xx^\top x - \gamma \nabla R(x)\|_F \leq O(\epsilon)$.
  i.e., $xx^\top \approx zz^\top$ under "scaling".
  x could be 0.



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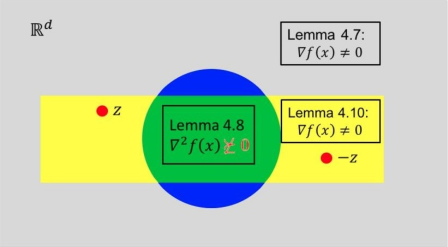
- $\|x\|^2 \geq \frac{1}{8}$. i.e., outside blue



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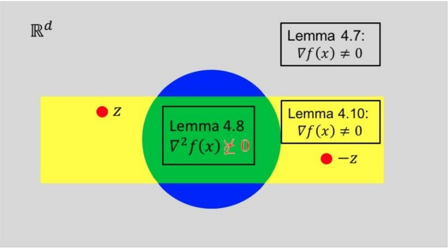
- $\|xx^\top - zz^\top\|_F^2 \leq O(\epsilon)$.



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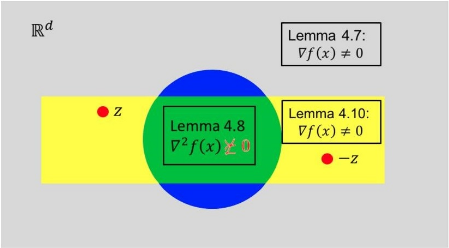

Combining them together:

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Handy Lemma:
If $\|xx^\top - zz^\top\|_F^2 \leq O(\epsilon)$, then $xx^\top = zz^\top = M$. [Sun and Luo 2015]

First thing first: optimality conditions

$$
\begin{aligned}
& f(x+\delta) \\
=& \frac{1}{2}\|P_\Omega(M-(x+\delta)(x+\delta)^\top\|_F^2 + \lambda R(x+\delta) + o(\|\delta\|^2) \\
=& \frac{1}{2}\|P_\Omega(M-xx^\top-(x\delta^\top+\delta x^\top)-\delta\delta^\top\|_F^2 \\
& + \lambda\left(R(x)+\langle\nabla R(x),\delta\rangle+\frac{1}{2}\delta^\top\nabla^2 R(x)\delta\right)+o(\|\delta\|^2) \\
=& f(x) - \langle P_\Omega(M-xx^\top), x\delta^\top+\delta x^\top\rangle + \langle\lambda\nabla R(x),\delta\rangle + o(\|\delta\|^2) \\
& -\langle P_\Omega(M-xx^\top),\delta\delta^\top\rangle + \frac{1}{2}\|P_\Omega(x\delta^\top+\delta x^\top)\|_F^2 + \frac{1}{2}\lambda\delta\nabla^2 r(x)\delta
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=& f(x) - \langle P_\Omega(M - xx^\top), x\delta^\top + \delta x^\top\rangle + \langle \lambda \nabla R(x), \delta\rangle + o(\|\delta\|^2) \\
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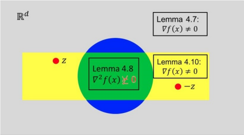
- First order condition: $2P_\Omega(M - xx^\top)x = \lambda \nabla R(x)$
- Second order condition:
  $\forall \delta, \|P_\Omega(\delta x^\top + x\delta^\top)\|_F^2 + \lambda \delta^\top \nabla^2 R(x)\delta \geq 2\delta^\top P_\Omega(M - xx^\top)\delta$

First order condition: yellow box



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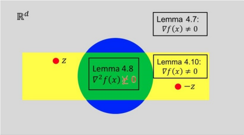

Theorem

Let $p \geq \frac{c \log d}{d}, \alpha = 10\mu/\sqrt{d}$, $\lambda \geq \mu^2 p/\alpha^2$. *Whp over* Ω, *for any* x *s.t.* $\nabla f(x) = 0$, *we have*

$$\|x\|_\infty \leq 4 \max\{\alpha, \mu\sqrt{p/\lambda}\} = 4\alpha$$

$R(x)$ helps in this case.

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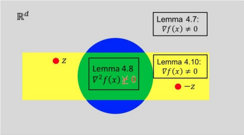

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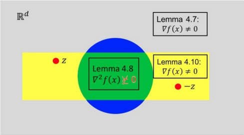

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- $i^* = \max_i |x_i|$. WLOG, $x_{i^*} \geq 0$.
- If $x_{i^*} \leq 2\alpha$, done. Otherwise $x_{i^*} > 2\alpha$

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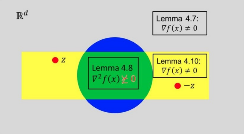

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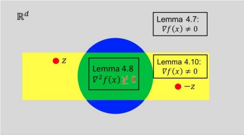
First order condition: yellow box



Theorem

Let $p \geq \frac{c \log d}{d}, \alpha = 10\mu/\sqrt{d}$, $\lambda \geq \mu^2 p/\alpha^2$. *Whp over* Ω, *for any* x *s.t.* $\nabla f(x) = 0$, *we have*

$$\|x\|_\infty \leq 4 \max\{\alpha, \mu\sqrt{p/\lambda}\} = 4\alpha$$

$R(x)$ helps in this case.

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$$\leq |x_{i^*}| \sum_{j \in S_{i^*}} |z_{i^*} z_j| \leq |x_{i^*}| \cdot \mu^2/d \cdot |S_{i^*}| \leq 2|x_{i^*}| p\mu^2$$

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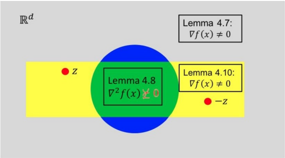
So $4|x_{i^*}|p\mu^2 \geq 2(P_\Omega(M - xx^\top)x)_{i^*} = (\lambda \nabla R(x))_{i^*} \geq \frac{\lambda}{2}|\mathbf{x}_{\mathbf{i}^*}|^3$
Therefore $|x_{i^*}| \leq 4\sqrt{p\mu^2/\lambda} \leq 4\alpha$.

Second order condition: outside blue ball



Theorem

Whp over Ω, for any $x \in$ yellow s.t. $\nabla^2 f(x) \succeq 0$, we have $\|x\|^2 \geq 1/8$.

Bound $R(x)$ in this case.

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- $\|x\|_\infty \leq 4\alpha, \|z_J\|_\infty \leq \frac{\mu}{\sqrt{d}}, \|z_J\| \leq 1$

Second order condition: outside blue ball

- **Linear property**:
  $\|P_\Omega(z_J x^\top + x z_J^\top)\|_F^2 = 2\langle P_\Omega(z_J x^\top), P_\Omega(z_J x^\top)\rangle + 2\langle P_\Omega(z_J x^\top), P_\Omega(x z_J^\top)\rangle \approx 2p\|x\|^2\|z_J\|^2 + 2p\langle x, z_J\rangle^2$

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- Therefore $\|x\|^2 \geq \frac{1}{3} \cdot \frac{1}{2} > \frac{1}{8}$

First order condition: $x \approx z$ after scaling

Theorem

Whp over Ω, for any x s.t. $\nabla f(x) = 0$, we have

$$\|zz^\top x - xx^\top x - \frac{\lambda}{2p} \cdot \nabla R(x)\| \leq O(\epsilon)$$

where $\epsilon = \tilde{O}(\mu^6 (pd)^{-1/2})$.
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Safely ignore $\nabla R(x)$..

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Whp over Ω, Suppose $x \in$ yellow s.t. $\|x\|^2 \geq \frac{1}{8}$, and $\nabla f(x) = 0$, we have $\|zz^\top x - xx^\top x\| \leq O(\epsilon)$.

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- For $i \in J^c$, we want $(z_i z^\top x - x_i x^\top x)^2 \leq (z_i z^\top x - x_i x^\top x - (\gamma \nabla R(x))_i)^2$
- I.e., $(z_i z^\top x - x_i x^\top x)(\nabla R(x))_i \leq 0$
- By definition, this is true because
  - $(\nabla R(x))_i = \gamma x_i$
  - $\|x\|^2 \geq \frac{1}{8}$
  - $|x_i| \geq \alpha \geq 4|z_i|$

Safely ignore $\nabla R(x)$..

Theorem

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Combine everything

Theorem

Suppose x s.t. $\|x\|^2 \geq \frac{1}{8}$, and $\|zz^\top x - xx^\top x\| \leq \delta$. Then for $\delta \in (0, 0.1)$, $\|xx^\top - zz^\top\|_F^2 \leq O(\delta)$.

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- $z = ux + v$, where $\langle v, x \rangle = 0$.

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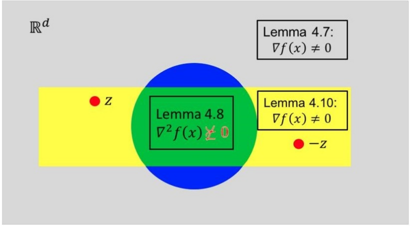
- So $|1 - u^2| \leq 8\delta$, $u\|v\| \leq 8\delta$.
- $|u| \approx 1, \|v\| \approx 0 \Rightarrow z \approx x$.

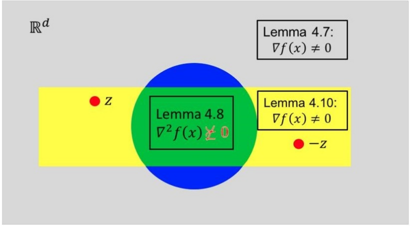
Rank k case...



$\mathbb{R}^d$
Lemma 4.7:
$\nabla f(x) \neq 0$
z
Lemma 4.8
$\nabla^2 f(x) \not\succeq 0$
Lemma 4.10:
$\nabla f(x) \neq 0$
$-z$

Rank k case...

$\nabla f(x) = 0 \Rightarrow \nabla f(X) = 0$



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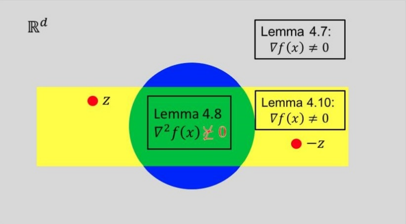

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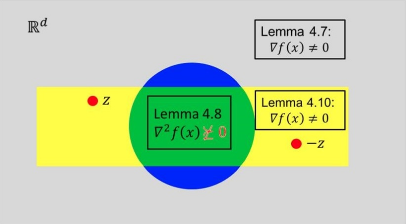

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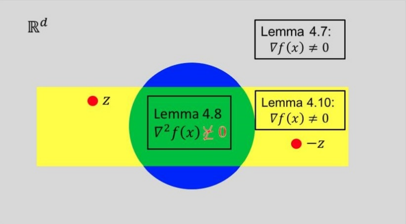

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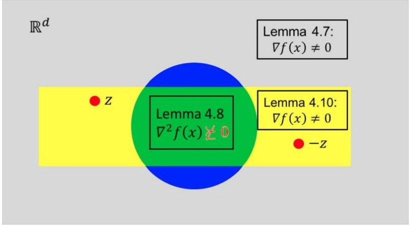

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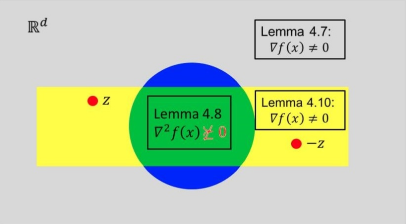

Combining them together:

- $\|xx^\top - zz^\top\|_F^2 \leq O(\epsilon) \Rightarrow \|XX^\top - ZZ^\top\|_F^2 \leq O(\epsilon)$

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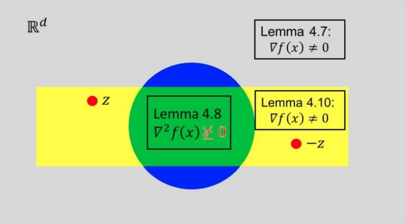

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Handy Lemma:

If $\|XX^\top - ZZ^\top\|_F^2 \leq O(\epsilon)$, $XX^\top = ZZ^\top = M$. [Sun and Luo 2015]

The lemma from Luo and Sun

Theorem (Locally one point strongly convex)

Suppose $p \geq O(\frac{\log d}{d})$, *whp over* Ω, *for any point* X *in* $B_\epsilon = \{X \in \mathbb{R}^{d \times r} : \|XX^\top - ZZ^\top\|_F \leq \epsilon\}$, *there exists* U *such that* $UU^\top = ZZ^\top$, *and*

$$\langle -\nabla f(X), U - X \rangle \geq \frac{p}{4} \|M - XX^\top\|_F^2$$

The lemma from Luo and Sun

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  - Slightly different, here it's using function value $\|M - XX^\top\|_F^2$.
  - Instead of $\|x_t - x^*\|_2^2$.
- If $\nabla f(X) = 0$, we have

$$0 = \langle -\nabla f(X), U - X\rangle \geq \frac{p}{4}\|M - XX^\top\|_F^2$$

So $XX^\top = M$.

Proof step 1

Theorem

If $\|XX^\top - M\|_F = \epsilon$, *there exists* U *such that* $UU^\top = M$ *and* $\|X - U\|_F \leq O(\epsilon\sqrt{r})$

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- $PP^\top = V(V^\top (M')^{-1} V)^{-1} V^\top = M'$

Proof step 2

- $\langle -\nabla f(X), U - X\rangle =$
  $\langle -\nabla \left(\frac{1}{2}\|P_\Omega(M - XX^\top)\|_F^2\right) - \nabla R(x), U - X\rangle$

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$$\langle -\nabla\left(\frac{1}{2}\|P_\Omega(M - XX^\top)\|_F^2\right), U - X\rangle = 2\langle P_\Omega(XX^\top - M)X, X - U\rangle$$

$$= \langle P_\Omega(XX^\top - M), (X - U)X^\top + X(X - U)^\top\rangle$$

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- $XX^\top - M = a + b$, $(X - U)X^\top + X(X - U)^\top = a + 2b$.

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Intuitively, a is first order, b is second order.

$$\langle P_\Omega(XX^\top - M), (X - U)X^\top + X(X - U)^\top\rangle$$

$$\geq \|P_\Omega(a)\|_F^2 + \|P_\Omega(b)\|_F^2 - 3\|P_\Omega(a)\|_F\|P_\Omega(b)\|_F \geq \frac{p}{4}\|M - XX^\top\|_F^2$$